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Solutions Approximating Solute Transport in a Leaky Aquifer Receiving Wastewater Injection

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16. ABSTRACT
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Solutions Approximating Solute Transport in a Leaky Aquifer Receiving Wastewater Injection

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A mathematical model amenable to analytical solution techniques is developed for the investigation of contaminant transport from an injection well into a leaky aquifer system, which comprises a pumped and an unpumped aquifer connected to each other by an aquitard. A steady state groundwater flow field is assumed, where the injected fluids move horizontally in the pumped aquifer and vertically in the aquitard. The unpumped aquifer is assumed to remain in hydrostatic condition due to its large transmissivity. Descriptions of groundwater velocities are based on appropriate leaky aquifer well hydraulics. The model assumes that contaminants are transmitted in the pumped aquifer by radial advection, and in the aquitard by vertical one-dimensional advection and longitudinal dispersion. Two coupled linear differential equations are formulated to deal with this transport problem; they involve variable coefficients dependent on Bessel functions. The unpumped aquifer is incorporated into the model as a boundary condition. Semianalytical transient solutions for concentration distributions in the pumped aquifer and the aquitard can be obtained by numerically inverting the Laplace domain solutions of the model with the *Crump* [1976] method. However, the steady state solutions of the model are determined in closed forms. In general, the modeling approach neglecting longitudinal dispersion in the pumped aquifer and lateral dispersion in the aquitard does not introduce significant error to the modelling results, as shown by the good agreement between numerical solutions including these two transport mechanisms and the solutions of the proposed model. Within the context of the assumed hydrogeological conditions, the proposed model appears to be a valid approximation to the problem of interest. Type curves for concentration evaluated at the well bore and the top of the aquitard are given, which can be employed to estimate whether the unpumped aquifer would be polluted as the consequence of the injection. Neglecting the leakage effects in a leaky aquifer can cause significant error in studying the transport problem.

INTRODUCTION

Injection into aquifers is one of the commonly used methods of wastewater disposal. One drawback of this practice is its potential of seriously polluting regional groundwater resources. Problems of solute transport from injection wells into groundwaters have been extensively studied, and a number of exact or approximate analytical solutions obtained. *Ogata* [1958], *Lau et al.* [1959], *Raimondi et al.* [1959], *Gelhar and Collins* [1971], *Dagan* [1971], *Tang and Babu* [1979], *Hsieh* [1986], and *Chen* [1987] gave solutions for problems involving a single injection well. *Hoopes and Harleman* [1967] and *Eldor and Dagan* [1972] developed solutions for problems involving a well doublet. All these solutions are exclusively related to completely confined aquifer conditions in which neither groundwater nor contaminants can be transmitted from the aquifer into the confining layers. Their descriptions of groundwater movement are based on the following relationship:

$$u'(r) = \frac{Q}{2\pi b_1 n_1 r} \tag{1}$$

Using the double-porosity approach, *Chen* [1985, 1986], *Feenstra et al.* [1984], *Barker* [1982], and *Hodgkinson and Lever* [1983] gave analytical or semianalytical solutions to wastewater injection into a single fracture situated in porous rock. Because most rocks have very low permeabilities, the injected fluids are assumed to move strictly within the fracture. However, the contained contaminants can be diffused to the surrounding rock due to the possible concentration gradient across the interface between the fracture and the porous matrix. These solutions are also applicable to problems pertaining to analogous injection in highly structured granular aquifers, provided that the fracture is treated as the aquifer and the porous matrix as the aquicludes. In such cases the groundwater flow fields are still describable by (1), for no groundwater movement can take place from the aquifer (the fracture) to the aquicludes (the porous matrix).

Under field conditions, however, many aquifers are bounded by low-permeability aquitards into which groundwater can still permeate. Such aquifers are commonly termed as "leaky" aquifers. In reality, a leaky aquifer is often just a part of a multiple-aquifer system; that is, it is often hydraulically connected with other aquifers via the overlying and underlying aquitards and thus pumpage from one affects the groundwater pressure in the others. As shown in Figure 1, wastewaters injected into such a hydrogeologic system primarily move through the pumped aquifer receiving injected fluids directly from the well. A portion of the wastewater, however, is able to transmit to the aquitards and may eventually reach the unpumped aquifer. Since the hydrogeological characteristics of the aquitard and the unpumped aquifer may significantly affect the groundwater movement in the pumped aquifer, the impact assessment of wastewater injection into a leaky aquifer should consider both the pumped and unpumped aquifers along with the connecting aquitards as one integral system (the leaky aquifer system). Within such a system the groundwater flow field involves "at least" two parts; one pertains to the groundwater movement in the pumped aquifer and the other to that in the aquitards. They are influenced by each other. Clearly, (1) fails to account for such a complex groundwater flow field, and therefore all these aforementioned solutions are incapable of dealing with wastewater injection into a leaky aquifer, which is of broad practical importance.

In this paper analytical and semianalytical solutions are

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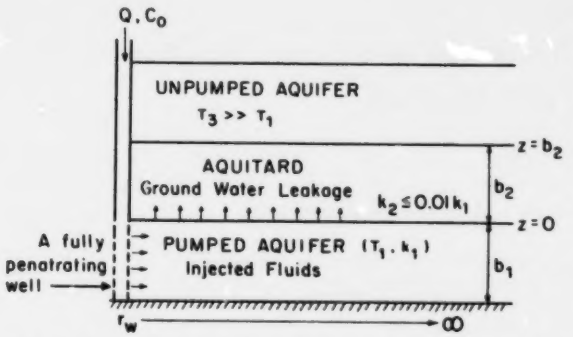

Fig. 1. Schematics of a leaky aquifer system for the current study.

determined for a model approximating solute transport from an injection well into a leaky aquifer system. The mathematical model is developed for the stated problem with an emphasis on the contaminant movement in the pumped aquifer and in the aquitard. In these two layers, groundwater flow fields are assumed to be steady state and quasi three-dimensional (i.e., horizontal and radial in the pumped aquifer and vertical in the aquitard). Their descriptions are primarily based on the well hydraulics theories given by *De Glee* [1930] or *Jacob* [1946]. The unpumped aquifer is assumed to be under hydrostatic conditions and is incorporated into the model as a constant head boundary condition. The injected solutes are assumed to be transported by horizontal advection only in the pumped aquifer while by vertical advection and the accompanied longitudinal mechanical dispersion in the aquitard; validity of such an approach is justified in this paper. The governing equations are formulated in two coupled linear differential equations which have variable coefficients dependent on Bessel functions. Transient solutions are obtained by using the Crump method [*Crump*, 1976] to numerically invert the Laplace domain solutions of the model. Closed-form solutions, however, are able to be determined for the steady state conditions. Specifically, the purposes of this paper are (1) to develop a mathematical model that is amenable to analytical or semianalytical solution techniques and can be used to quantify the solute movement from an injection well into a hydraulically connected aquifer and aquitard and (2) to determine the transient and steady state solutions for the proposed model. As a first approximation, these solutions provide simplified yet reasonable quantitative means to deal with the complicated problem of wastewater injection into a leaky aquifer.

MATHEMATICAL MODEL

Groundwater Flow Field

For the proposed transport model, the groundwater movement in the leaky aquifer system is simulated by invoking the following three major assumptions.

1. Groundwater flows horizontally (and radially symmetric with respect to the well) in the pumped aquifer, while vertically in the aquitard. By the tangent law, this orthogonal refraction of flow lines is justifiable if the hydraulic conductivity of the pumped aquifer is at least two orders of magnitude greater than that of the aquitard (i.e., $k_1 > 100\ k_2$). Such a hydraulic conductivity contrast is not unusual to observe in practice. *Neuman and Witherspoon* [1969*a*, *b*] indicated that the errors involved in this quasi three-dimensional modeling approach are usually less than 5%, provided $k_1 > 100\ k_2$. As a matter of fact, assumption 1 has been widely adopted in many analytical or numerical modeling works. Recent advancement in analytical solutions by *Chen et al.* [1986], however, considered two-dimensional flow in the aquitard.

2. The potentiometric field in the unpumped aquifer is not disturbed by the pumping (the unpumped aquifer is simply assumed to remain in hydrostatic conditions during the pumping period). Basically, this assumption is valid only for short pumping periods, during which the pressure perturbation due to injection has not reached the unpumped aquifer. However, as discussed by *Neuman and Witherspoon* [1969*a*, *b*], if the transmissivity of the unpumped aquifer is at least two orders of magnitude greater than that of the pumped aquifer (i.e., $T_3 > 100\ T_1$), effects of possible head buildup in the unpumped aquifer on the rest of the system may be neglected. That is, this particular assumption is generally valid if $T_3 > 100\ T_1$. Under field conditions, this assumption is probably easier to justify if the unpumped aquifer is actually a lake, a ponding water body, or even a saturated coarse-grained geological medium.

3. Steady state conditions for groundwater movement have been reached in the pumped aquifer as well as in the aquitard (i.e., storage effects are negligible in both layers). The storage effect is a time-dependent process; its importance diminishes as injection proceeds, eventually vanishing under steady state conditions. *Bredehoeft and Pinder* [1970] found that the storage effect in the aquitard became negligible if

$$t > \frac{s_2}{10}\frac{b_2}{k_2} \tag{2}$$

where s_2, b_2, and k_2 are the storage coefficient, the uniform thickness and the hydraulic conductivity of the aquitard, respectively. It should be noted that steady state conditions can occur only if the head in the unpumped aquifer remains constant, a condition set forth in assumption (2).

Although (2) predicts the development time for steady state conditions in the aquitard only, it may be used to estimate the "minimum" time requirement for steady state conditions in both the pumped aquifer and the aquitard when no other relationship is available. In reality, the actual time for assumption (3) to be valid may require longer injection periods than the estimated by (2).

In addition to the three major assumptions, each layer in the leaky aquifer system is assumed to be homogeneous, isotropic and of a uniform thickness. The injection well fully penetrates the pumped aquifer only, and discharges fluids at a constant flow rate Q. As a result, the steady state head buildup distributions in the pumped aquifer can be appropriately described by the solution given by *Jacob* [1946] or *De Glee* [1930] as

$$h_1 = \frac{Q}{2\pi T_1} K_0\left(\frac{r}{B}\right) \tag{3}$$

where $K_0(\ )$ is the modified Bessel function of the second kind of order zero and B is the leakage factor defined as

$$B = \left(\frac{T_1}{k_2/b_2}\right)^{1/2} \tag{4}$$

By applying Darcy's Law to (3) the steady state groundwater seepage velocity in the pumped aquifer is found to be

$$u = \frac{-k_1}{n_1}\frac{dh_1}{dr} = \frac{Q}{2\pi n_1 b_1 B} K_1\left(\frac{r}{B}\right) \tag{5}$$

where $K_1(\ )$ is the modified Bessel function of the second kind of order one, and the recurrence formula for derivatives of Bessel functions is used (see equation 9.6.28 of *Abramowitz and Stegun* [1970]). When $x > 0$, both $K_0(x)$ and $K_1(x)$ are positive, monotonically decreasing, and $K_1(x) \geq K_0(x)$.

Under the assumed conditions, the vertical groundwater leakage velocity is linearly proportional to the head losses across the aquitard thickness. Since the unpumped aquifer has been assumed to be in hydrostatic conditions the head at the top of the aquitard, $h_2(b_2)$, can be set to zero by considering the continuity of head distributions at the interface of the two layers. Therefore the vertical seepage velocity in the aquitard is

$$v = \frac{-k_2}{n_2}\left(\frac{h_2(b_2) - h_1}{b_2}\right) = \frac{Q}{2\pi n_2 B^2} K_0\left(\frac{r}{B}\right) \tag{6}$$

Equation (6) indicates that the vertical groundwater leakage velocity is constant across the aquitard at any specific r, while its magnitude decreases as r increases. Hence (5) and (6) completely describe the groundwater flow field for the proposed model.

As shown by (4), the leakage factor essentially represents the ratio of the pumped aquifer transmissivity T_1 to the aquitard leakance, k_2/b_2. A large B may be due to a large T_1 and/or a small k_2/b_2; in either case the pumped aquifer dominates the transmission of the injected fluids. Indeed, the groundwater movement from a well into a leaky aquifer having a large B is similar to that in a confined aquifer. Mathematically, this can be explained by noting that if B is large then $K_1(r/B)$ and $K_0(r/B)$ can be approximated by [e.g., *Hantush*, 1964]

$$K_1\left(\frac{r}{B}\right) \simeq \frac{B}{r} \qquad \frac{r}{B} \leq 0.05 \tag{7a}$$

$$K_0\left(\frac{r}{B}\right) \simeq \ln\left(1.12\,\frac{B}{r}\right) \qquad \frac{r}{B} \leq 0.05 \tag{7b}$$

Using (7*a*) in (5) renders u become identical to u' as depicted in (1), which is valid for confined conditions. In accordance to (7*b*) and (6), the associated groundwater leakage velocity is

$$v = \frac{Q}{2\pi n_2 B^2} \ln\left(1.12\,\frac{B}{r}\right) \tag{8}$$

which is small, since magnitudes of $\ln(1.12\ B/r)$ do not override the smallness of B^{-2} when B is large. This is in agreement with the discussion that the leaky aquifer having a large B reacts to pumping as a confined aquifer; the same discussion was made by *Neuman and Witherspoon* [1972]. In addition to the hydrogeologic parameter B, the pumping rate Q is an important hydraulic factor that can also affect the leakage condition of an aquifer. If the head buildup in the pumped aquifer caused by a small injection rate is not sufficient to move groundwater into the aquitard, the pumped aquifer performs like a confined one even it is subject to a small leakage factor.

Equation (7) can be used to approximate u and v near the well not because of large B's but because of small r's. When r is small, the groundwater leakage velocity is not necessarily negligible although u can be identical to u'. This means that in a leaky aquifer where the leakage factor is not large, the groundwater movement in the vicinity of a pumping well (e.g., $r < 0.05B$) remains similar to that in a confined aquifer, while noticeable groundwater leakage to the aquitard may exist (i.e., v may not be negligibly small).

The mathematical model for the transport problem of interest will be developed based on the groundwater flow conditions given by (5) and (6). In this model, the solute concentrations in the pumped aquifer C_1 are considered to be vertically averaged over the pumped aquifer thickness. Hence vertical transport in the pumped aquifer is not taken into account. Injected solutes are assumed to be transported by horizontal advection in the pumped aquifer, while by vertical advection and mechanical dispersion in the aquitard. Molecular diffusion is neglected in both layers. Possible simple attenuation mechanisms such as linear equilibrium adsorption and first order radioactive (or biological) decay are taken into account.

Transport Equations in the Pumped Aquifer

In the following development, both advection and longitudinal mechanical dispersion (although it is not considered in the model) are included to yield a generalized "advection-dispersion equation" for the pumped aquifer. From it the "governing equation" of the proposed model for the pumped aquifer is derived by omitting terms related to longitudinal dispersion. Results from these two equations will be used to justify the neglect of longitudinal dispersion in the pumped aquifer.

By applying the mass balance principle to a control volume in the pumped aquifer, the advection-dispersion equation in this layer can be expressed as

$$ud_1\frac{\partial^2 C_1}{\partial r^2} + \left[\frac{d_1}{r}\frac{\partial(ur)}{\partial r} - u\right]\frac{\partial C_1}{\partial r} - \frac{1}{r}\frac{\partial(ur)}{\partial r}C_1 - \frac{n_2}{n_1 b_1}\left(vC_2 - D_2\frac{\partial C_2}{\partial z}\right)_{z=0} = \frac{\partial C_1}{\partial t} \tag{9}$$

where C_1 and C_2 are the concentrations in the pumped aquifer and in the aquitard, respectively, and d_1 is the constant longitudinal dispersivity of the pumped aquifer. Denoted by D_2, the dispersion coefficient neglecting the diffusion coefficient in the aquitard is

$$D_2 = vd_2 \tag{10}$$

where d_2 is the constant longitudinal dispersivity of the aquitard.

The fourth term of (9) accounts for the solute mass flux associated with the groundwater leakage. At the interface between the pumped aquifer and the aquitard (i.e., $z = 0$), C_1 is assumed to be equal to C_2 so that vC_2 in the fourth term of (9) can also be expressed as vC_1. By noting (5) and (6), it is found that

$$\frac{1}{r}\frac{\partial(ur)}{\partial r} = \frac{-n_2}{n_1 b_1} v \tag{11}$$

where the same recurrence formula used in obtaining (5) is used. Introducing (11) to (9), (9) can be simplified to a more compact form as

$$ud_1\frac{\partial^2 C_1}{\partial r^2} - \left(\frac{n_2 d_1}{n_1 b_1} v + u\right)\frac{\partial C_1}{\partial r} + \frac{n_2 D_2}{n_1 b_1}\left.\frac{\partial C_2}{\partial z}\right|_{z=0} = \frac{\partial C_1}{\partial t} \tag{12}$$

Equation (12) is the advection-dispersion equation for

solute transport in the pumped aquifer without taking into consideration the linear equilibrium adsorption and radioactive decay processes. Mathematical representation of these two processes are not affected by the relationships describing the groundwater flow fields. Thus they can be included in (12) by making use of the retardation factor and the decay constant that are normally employed in modeling solute transport in nonleaky aquifers. In a manner similar to that used by *Chen* [1986], the advection-dispersion equation for the pumped aquifer including radioactive decay and linear isotherm adsorption is found to be

$$ud_1 \frac{\partial^2 C_1}{\partial r^2} - \left(\frac{n_2 d_1}{n_1 b_1} v + u\right) \frac{\partial C_1}{\partial r} + \frac{n_2 D_2}{n_1 b_1} \frac{\partial C_2}{\partial z}\bigg|_{z=0} - \lambda R_1 C_1 = R_1 \frac{\partial C_1}{\partial t} \qquad (13)$$

where λ is the decay constant for the possible radioactive decay or biological processes, and R_1 is the dimensionless retardation factor in the pumped aquifer for linear isotherm adsorption (see, for example, *Freeze and Cheery* [1979]). It is important to note that u, v, and D_2 are actually dependent on the Bessel functions $K_0(r/B)$ or $K_1(r/B)$.

In fact, (11) itself represents the principle of groundwater conservaton. This is manifested by writing (11) in its equivalent form of

$$\frac{\partial}{\partial r}(2\pi n_1 b_1 r u) = -2\pi n_2 r v \qquad (14)$$

Considering a cylinder with a radius r as an annular control volume having an infinitesimal radial thickness, the left-hand side of (14) represents changing of groundwater flow rates across the cylindrical surface area, being normal to u, in the pumped aquifer. The right-hand side of (14) reflects the amount of groundwater emitted from the top perimeter of this control volume to the adjacent aquitard. Since steady state flow conditions prevail (i.e., no change of groundwater mass inside this control volume), these two parts must be equal; a negative sign indicates that decrease of groundwater flow rates in the pumped aquifer is attributed to increase of groundwater leakage to the aquitard.

If the pumped aquifer is completely nonleaky, neither groundwater nor contaminants can be transmitted to the aquitard. This can be appropriately achieved in (13) by setting v and n_2 equal to zero. Correspondingly, the groundwater velocity in the pumped aquifer should be described by u' instead of u. As a result, (13) is reduced to the well-known radial dispersion equation for a confined aquifer. If the pumped aquifer is considered as a single fracture contained in porous rock, n_2 is not zero while v is. In this event, however, molecular diffusion from the fracture to the porous rock needs to be considered. To do so, D_2 is replaced by the effective molecular diffusion coefficient of the porous rock, thereby converting the third term in (13) to a pure diffusion flux evaluated at $z = 0$. Again u now should be replaced by u'. Consequently, (13) becomes identical to the radial dispersion equation for a fracture flow problem [see *Chen*, 1986].

Since the proposed model neglects longitudinal dispersion in the pumped aquifer, the desired governing equation for the pumped aquifer can be derived from (13) by setting d_1 equal to zero as

$$-u \frac{\partial C_1}{\partial r} + \frac{n_2 D_2}{n_1 b_1} \frac{\partial C_2}{\partial z}\bigg|_{z=0} - \lambda R_1 C_1 = R_1 \frac{\partial C_1}{\partial t} \qquad (15)$$

which will be used in determining the solutions of the stated problem.

Transport Equation in the Aquitard

In simulating the transport problems accompanied with the groundwater leakage described by (6), the aquitard is conceptualized as a set of infinite concentric cylinders filled with identical porous material. In each of the cylinders, the groundwater velocity is uniform (i.e., v is constant across the aquitard at a specific r). However, the groundwater velocities in cylinders located at different r' are varied in accordance to (6). Contaminants emitted from the top of the pumped aquifer are assumed to move in each cylinder by means of one-dimensional vertical advection and the accompanied longitudinal mechanical dispersion. The dispersion coefficient D_2, being defined in (10), is constant within a cylinder (yet its magnitude is different in cylinders located at different r' due to the changing v). In essence, the transport phenomena in each cylinder is studied as a one-dimensional dispersion problem in a uniform groundwater flow field. Mass transfer among any adjacent cylinders is not taken into account; validity of this assumption will be discussed below. Using these assumptions and considering linear isotherm adsorption and radioactive (or biological) decay gives the governing equation in the aquitard as

$$D_2 \frac{\partial^2 C_2}{\partial z^2} - v \frac{\partial C_2}{\partial z} - \lambda R_2 C_2 = R_2 \frac{\partial C_2}{\partial t} \qquad (16)$$

where R_2 is the dimensionless retardation factor in the aquitard. Although D_2 and v can vary with r, they are invariant of the independent spatial and temporal variables, z and t, in (16). Therefore D_2 and v are treated as constant coefficients in (16), rendering it mathematically identical to the well-known one-dimensional dispersion equation for a uniform groundwater flow field. Equation (16) is coupled with (15) through the second term of (15) which pertains to the gradient of C_2 on the interface at $z = 0$.

Contaminants leaked into the aquitard may eventually reach the unpumped aquifer. A transport equation can be developed for the contaminant migration in the unpumped aquifer. However, this would broaden the mathematical model to include three simultaneous differential equations, for which analytical or semianalytical solutions are difficult to obtain. Since detailed investigation of contaminant movement in the unpumped aquifer is not of primary interest in this study, no governing for this layer is formulated and the coupled (15) and (16) constitute the complete governing equations of the proposed model. The unpumped aquifer plays a role as a boundary condition to (16). Different solutions to (15) and (16) can be obtained if different appropriate initial and boundary conditions are prescribed. In the current study, only one set of boundary conditions is employed to determine the solutions, as discussed below.

SEMIANALYTICAL AND ANALYTICAL SOLUTIONS

Recapitulating (15) and (16) in appropriate dimensionless form gives the dimensionless governing equations as

$$\frac{\partial^2 C_2}{\partial \xi^2} - 2\eta \frac{\partial C_2}{\partial \xi} - \frac{\theta}{K_0(\rho)} C_2 = \frac{\beta}{K_0(\rho)} \frac{\partial C_2}{\partial \tau} \qquad (17)$$

$$-K_1(\rho) \frac{\partial C_1}{\partial \rho} + \frac{K_0(\rho)}{2\eta} \frac{\partial C_2}{\partial \xi}\bigg|_{\xi=0} - \alpha C_1 = \frac{\partial C_1}{\partial \tau} \qquad (18)$$

where ξ is the dimensionless vertical distance in the aquitard defined by z/b_2, ρ is the dimensionless radial distance defined by r/B, τ is the dimensionless time, and η, θ, β, and α are dimensionless constant parameters with definitions in the Notation.

The initial and boundary conditions for (17) and (18) are prescribed as

$$C_1(\rho, 0) = C_2(\xi, 0) = 0 \tag{19}$$

$$C_1(\rho_0, \tau) = 1 \tag{20}$$

$$C_1(\rho \to \infty, \tau) = 0 \tag{21}$$

$$C_2(0, \tau) = C_1(\rho, \tau) \tag{22}$$

$$C_2(\xi \to \infty, \tau) = 0 \tag{23}$$

The initial condition of (19) indicates that the system involves no contaminants prior to injection. After injection starts, the injected fluids are assumed to be well mixed at a constant concentration inside the well bore and are transmitted to the immediate surrounding pumped aquifer at this same concentration, as indicated by (20). In (22), a continuity of concentration is assumed at the interface between the pumped aquifer and the aquitard at $\xi = 0$ (i.e., $z = 0$). Despite the aquitard's finite thickness b_2 (such that $0 \le \xi \le 1$), the boundary condition (23) mathematically extends the aquitard thickness to infinity. This treatment is equivalent to assuming that contaminant transport occurring inside the unpumped aquifer does not affect the transport phenomena inside the aquitard. A similar approach has been used by *van Genuchten and Parker* [1984] to analyze solute transport in a finite system subject to a free dripping outlet. In addition, as discussed in the appendix, other possible outlet boundary conditions at b_2 such as $\partial C_2/\partial Z = 0$ (the well-known *Danckwerts* [1953] condition) and $C_2 = 0$ (an infinite dilution capacity in the unpumped aquifer) lead to solutions identical to that associated with (23) provided that $b_2 > 5.6\, d_2$, a condition that is often satisfied in the field. Therefore (23) is employed to find unique solutions for the proposed model with the understanding that practically meaningful values of C_2 exists only in the range of $0 \le \xi \le 1$.

Transient Solutions

Applying the Laplace transform with respect to τ to (17) through (23) gives

$$\frac{d^2G_2}{d\xi^2} - 2\eta \frac{dG_2}{d\xi} - \frac{\theta + p\beta}{K_0(\rho)} G_2 = 0 \tag{24}$$

$$-K_1(\rho) \frac{dG_1}{d\rho} + \frac{K_0(\rho)}{2\eta} \left.\frac{dG_2}{d\xi}\right|_{\xi=0} - (p + \alpha)G_1 = 0 \tag{25}$$

$$G_1(\rho_0, p) = 1/p \tag{26}$$

$$G_1(\rho \to \infty, p) = 0 \tag{27}$$

$$G_2(0, p) = G_1(\rho, p) \tag{28}$$

$$G_2(\xi \to \infty, p) = 0 \tag{29}$$

where

$$G_i = \int_0^\infty e^{-p\tau} C_i \, d\tau \qquad i = 1, 2 \qquad p \in C$$

Since $K_0(\rho)$ is invariant of the independent variable ξ, (24) can be treated as a second-order linear ordinary differential equation with "constant" coefficients. Hence the solution for (24) subject to (28) and (29) is

$$G_2 = G_1 \exp\{[\eta - q(\rho, p)]\xi\} \tag{30}$$

where

$$q = \left[\eta^2 + \frac{\theta + p\beta}{K_0(\rho)}\right]^{1/2} \tag{31}$$

Introducing the first derivative of G_2 evaluated at $\xi = 0$ to (25) yields

$$-K_1(\rho) \frac{dG_1}{d\rho} + \left(\frac{K_0(\rho)}{2} - \frac{K_0(\rho)\varsigma}{2\eta} - p - \alpha\right)G_1 = 0 \tag{32}$$

By integrating (32) and noting (26) and (27), the solution of G_1 can be determined as

$$G_1 = \frac{1}{p} \exp\left\{\int_{\rho_0}^{\rho} \left(\frac{K_0(x)}{2} - \frac{q(x, p)K_0(x)}{2\eta} - p - \alpha\right) \frac{dx}{K_1(x)}\right\} \tag{33}$$

The transient solutions for C_1 and C_2 are the Laplace inversion (with respect to p) of (33) and (30), respectively. Due to the difficulty in analytically inverting these equations, numerical inversion techniques are employed to find C_1 and C_2. Three methods as given by *Stehfest* [1970*a*, *b*], *Dubner and Abate* [1968] and *Crump* [1976] were tried. As shown in Figure 2(top), the Stehfest method yields oscillatory results for C_1 except for large τ, and thus it is not recommended for the current work. The other two methods give essentially the same smooth curves for both C_1 and C_2 for all τ (see Figures 2(top) and 2(bottom)). In essence, the Dubner and Abate method inverts the Laplace transform of a function by expressing the transformed function in a Fourier cosine series, which is computed with the fast Fourier transform methods (see, for example, *Cooley and Tukey* [1965]). The Crump method, however, approximates the Laplace inversion by writing the transformed function in a combined Fourier cosine and sine series, which is calculated with the aid of the epsilon algorithm [e.g., *MacDonald*, 1964]. The Laplace transform parameter p needs to be declared as complex in computation when these two methods are used. Since the Crump method is computationally more efficient (i.e., less CPU time), it is recommended for the determination of transient C_1 and C_2 which are termed semianalytical solutions here. Based on the Crump method, a FORTRAN subroutine called DINLAP [*International Mathematics and Statistics Library, Inc.*, 1987] is readily available for inverting (30) and (33). Integrals involved in these two equations can be numerically evaluated using Gaussian quadratures without difficulty.

Steady State Solutions

In a straightforward manner, the steady state solutions for the proposed model can be acquired by omitting the transformed parameter p in (33) and (30) as

$$C_1 = \exp\left\{\int_{\rho_0}^{\rho} \left(\frac{K_0(x)}{2} - \frac{q'(x)K_0(x)}{2\eta} - \alpha\right) \frac{dx}{K_1(x)}\right\} \tag{34}$$

$$C_2 = C_1 \exp\{[\eta - q'(\rho)]\xi\} \tag{35}$$

where

$$q'(\rho) = \left[\eta^2 + \frac{\theta}{K_0(\rho)}\right]^{1/2} \tag{36}$$

Under the given boundary conditions, nontrivial solutions

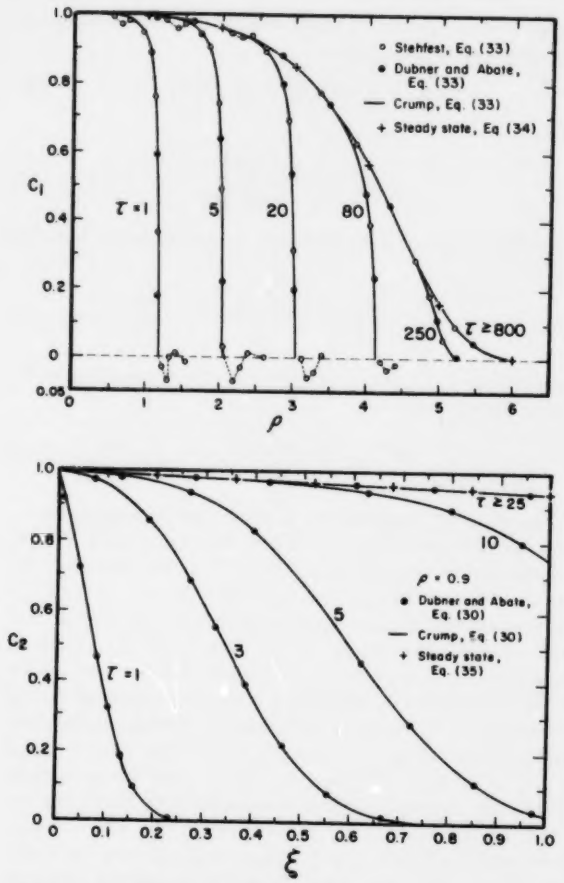

Fig. 2. Comparisons of semianalytical solutions of the proposed model by three numerical inversion methods, where $\eta = 12.5$, $\beta = 93.75$, $\theta = 0.7$, and $\alpha = 0.0075$: (top) C_1 versus ρ for various τ and (bottom) C_2 versus ξ for various τ at $\rho = 0.9$.

for (34) and (35) exist only if the decay process prevails (i.e., $\lambda \neq 0$). One of the nontrivial steady state solutions are presented in Figure 2. It is in excellent agreement with the numerical inversion of (30) and (33) at large time periods approaching the steady state conditions, supporting the validity of the semianalytical transient solutions. Should λ be zero, both θ and α become zero, yielding a trivial solution of unity for C_1 and C_2 for all ξ and ρ.

Results and Discussions

Neglect of Longitudinal Dispersion in the Pumped Aquifer

Only advection is taken into consideration as the transport mechanism in the pumped aquifer. If longitudinal dispersion is included, the advection-dispersion equation for movement of injected solutes in the pumped aquifer is (13), which can be expressed in a dimensionless form as

$$lK_1(\rho)\frac{\partial^2 C_1}{\partial \rho^2} - [lK_0(\rho) + K_1(\rho)]\frac{\partial C_1}{\partial \rho} + \frac{K_0(\rho)}{2\eta}\frac{\partial C_2}{\partial \xi}\bigg|_{\xi=0} - \alpha C_1 = \frac{\partial C_1}{\partial \tau} \qquad (37)$$

where l, defined by d_1/B, is the only parameter involving the longitudinal dispersivity. Equation (37) reduces to (18) when longitudinal dispersion is neglected by setting l equal to zero.

Due to the assumptions underlying the quasi-three-dimensional approach, the smallest permissible ratio of k_1 and k_2 is 100. If the maximum contrast between k_1 and k_2 is limited to four orders of magnitude (so that the "aquitard" does not function as an "aquiclude" in which groundwater is nearly immobile), the largest ratio of k_1 to k_2 is 10,000. Under these conditions, B ranges from $10(b_1b_2)^{1/2}$ to $100(b_1b_2)^{1/2}$, i.e., a few tens of meters to several thousands of meters (say, 20–5000 m) dependent on the magnitudes of b_1 and b_2. Laboratory and field experiments give values of d_1 ranging from a few centimeters to a few tens of meters, depending on the scale of the experiments. Correspondingly, l is generally small (less than unity). Due to the smallness of l, in (37) the terms multiplied by l can effectively to neglected in comparison to other terms independent of l. Hence the modeling approach ignoring longitudinal dispersion in the pumped aquifer appears to be reasonable.

To further examine the modeling approach neglecting longitudinal dispersion in the pumped aquifer, (37) and (17) are simultaneously solved by the numerical method of cubic spline collocation. It has been recently employed to solve boundary value problems in fluid mechanics [*Rubin and Graves*, 1975; *Rubin and Khosla*, 1976; *Wang and Kahawita*, 1983; *Tewarson and Zhang*, 1986]. In this method, the solution of the differential equation of interest is approximated as the cubic spline, which together with its first and second derivatives is continuous over a specific domain. By making use of these continuity properties, relationships designating the interrelations among the cubic spline function, its first and second derivatives are determined at nodal points in a discretized solution domain. Then these cubic spline relationships are incorporated into the solution procedures for differential equations. The cubic spline collocation method gives the approximation of the first derivatives a fourth-order accuracy in a uniform mesh and a third-order accuracy in a nonuniform mesh. The second derivative approximation is of second-order accuracy for both uniform and nonuniform grids. Mathematics and programing techniques for this method are straightforward; detailed discussions can be found in the paper by *Rubin and Graves* [1975]. Considering that the purposes of the current work are to discuss the proposed model and its solutions, any further development and illustration of the cubic spline models for verifications addressed here are not elucidated but are referred to in the report by C. S. Chen and G. He (unpublished manuscript, 1988).

In Figure 3, semianalytical solutions for C_1 obtained from (18) and (17) are compared with the numerical solutions of (37) and (17) for $l = 0.01$ and 0.001. Figure 3(top) refers to a condition without the decay process while Figure 3(bottom) to that with a decay process. Discrepancy between the semianalytical and numerical solutions is significant when $l = 0.01$ but is reduced as l decreases to 0.001. It was found that the convergence of the numerical solutions to the semianalytical solutions was not significantly improved by continuously decreasing the magnitudes of l, which may be due to the truncation/roundoff errors associated with the numerical scheme. Nevertheless, Figure 3 quantitatively demonstrates that terms involving l in (37) can be neglected without causing significant error if l is small (say, $l \leq 0.001$). This conclusion is also reached for various other sets of data; they are documented in

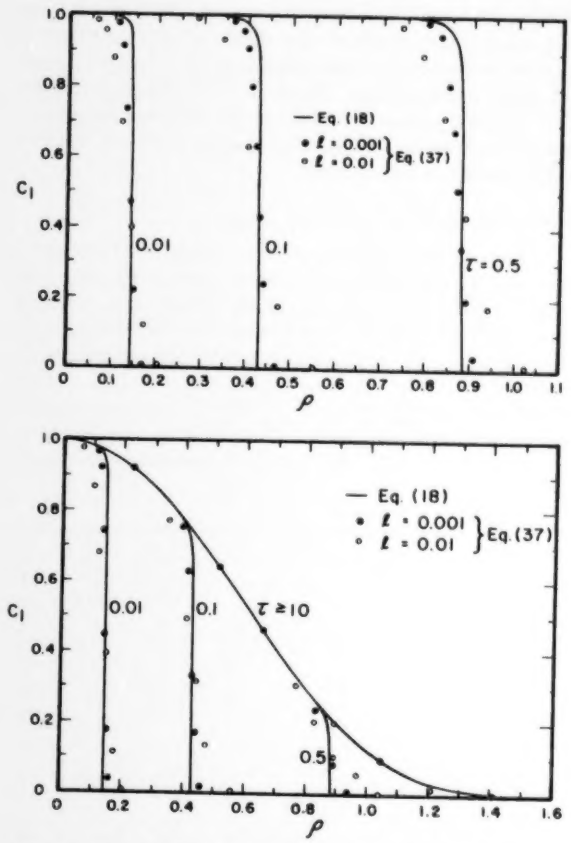

Fig. 3. Comparisons of semianalytical solutions of (18) subject to (17) with numerical solutions of (37) subject to (17) indicating the effects of neglecting longitudinal dispersion in the pumped aquifer ($\eta = 50$, $\beta = 150$): (top) no decay process with $\alpha = \theta = 0$ and (bottom) decay process exists with $\alpha = 3$ and $\theta = 450$.

the report by C. S. Chen and G. He (unpublished manuscript, 1988) and are not reproduced here. Therefore neglecting longitudinal dispersion in the pumped aquifer is not an unreasonable simplification for the problem under investigation, and errors introduced by it are negligible where l is less than or equal to 0.001, a condition not unusual in the field.

Neglect of Lateral Dispersion in the Aquitard

The proposed model assumes that solute transport in the aquitard is by one-dimensional vertical advection and longitudinal mechanical dispersion on the groundwater flow field illustrated by (6). It shows that $v(r)$ decreases rapidly in the radial direction. Accordingly, horizontal concentration gradients exist in the aquitard and may cause lateral dispersion in the radial direction that is not considered in the model. It is of practical interest to know if significant errors are generated by neglecting lateral dispersion in the aquitard.

Subject to the mass balance principle, adding lateral dispersion to (16) gives a two-dimensional transport equation as

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r(vd_2')\frac{\partial C_2}{\partial r}\right] + D_2\frac{\partial^2 C_2}{\partial z^2} - v\frac{\partial C_2}{\partial z} - \lambda R_2 C_2 = R_2\frac{\partial C_2}{\partial t} \qquad (38)$$

in which the first term represents the effects of lateral dispersion in the radial direction; the product of v and the aquitard lateral dispersivity d_2' denotes the lateral mechanical dispersion coefficient. The first term of (38) can be expanded by the chain rule and the appropriate recurrence formula for the differentiation of Bessel functions. Writing the expanded (38) in appropriate dimensionless form gives

$$\omega\frac{\partial^2 C_2}{\partial \rho^2} + \omega\left[\frac{1}{\rho} - \frac{K_1(\rho)}{K_0(\rho)}\right]\frac{\partial C_2}{\partial \rho} + \frac{\partial C_2}{\partial \xi^2} - 2\eta\frac{\partial C_2}{\partial \xi} - \frac{\theta}{K_0(\rho)}C_2 = \frac{\beta}{K_0(\rho)}\frac{\partial C_2}{\partial \tau} \qquad (39)$$

where ω is defined by $d_2'b_2^2/(d_2B^2)$. In general, the magnitude of ω is small due to the large B^2 being involved in the denominator of the definition. Therefore the first two terms associated with ω in (39) appear to be negligible without introducing significant error to the solution of C_2, supporting the modeling approach ignoring lateral dispersion in the aquitard.

To quantitatively justify the neglect of lateral dispersion in the aquitard, (39) and (37) have been solved with the aforementioned cubic spline collocation method and the numerical results compared to the semianalytical solutions. The mag-

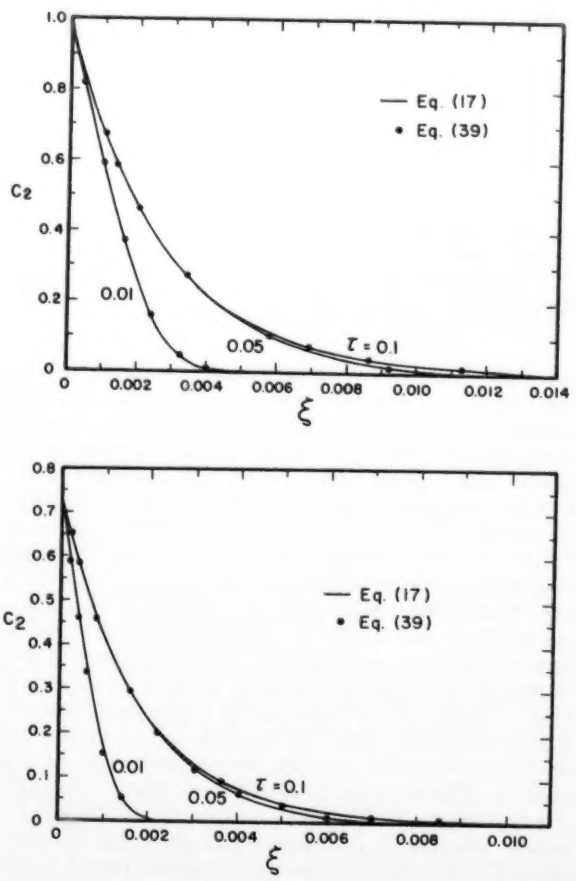

Fig. 4. Comparisons of semianalytical solutions of (17) subject to (18) with numerical solutions of (39) subject to (37) indicating the effects of neglecting lateral dispersion in the aquitard with the decay process as $\omega = 0.2$, $\eta = 1000$, $\beta = 6 \times 10^4$, $\alpha = 60$, and $\theta = 3.6 \times 10^6$. (Top) C_2 evaluated at $\rho = 0.02$ and (bottom) C_2 evaluated at $\rho = 0.1$.

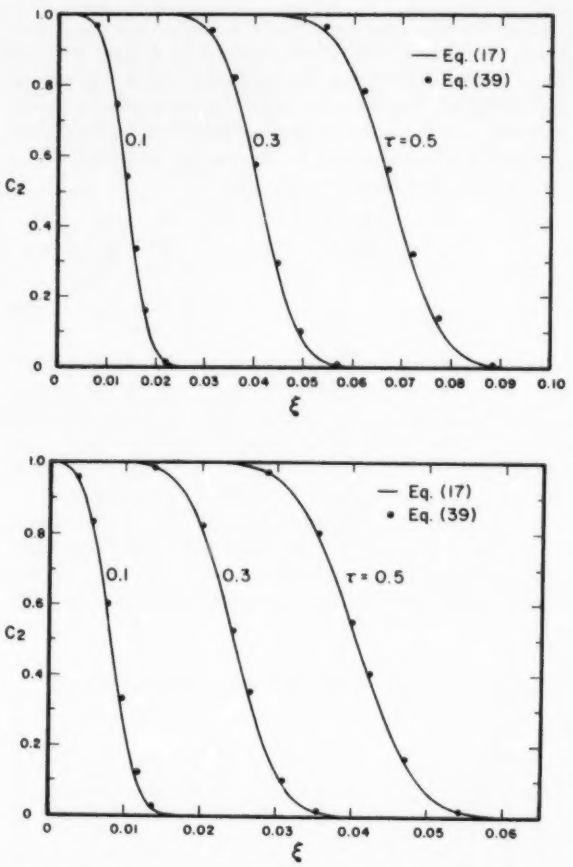

Fig. 5. Comparisons of semianalytical solutions of (17) subject to (18) with numerical solutions of (39) subject to (37) indicating the effects of neglecting lateral dispersion in the aquitard without the decay process as $\omega = 0.2$, $\eta = 1000$, $\beta = 6 \times 10^4$, and $\alpha = \theta = 0$. (Top) C_2 evaluated at $\rho = 0.02$ and (bottom) C_2 evaluated at $\rho = 0.1$.

nitude of l was set to be 0.001 when numerical solutions of (37) and (39) were determined. Comparisons of these numerical solutions to the semianalytical solutions of the proposed model thus can indicate (1) whether lateral dispersion in the aquitard can be neglected and (2) the appropriateness of neglecting longitudinal dispersion in the pumped aquifer with the presence of lateral dispersion in the aquitard. As shown in Figures 4–7, semianalytical solutions for C_1 and C_2 of the proposed model agree well with the numerical solutions of (37) and (39) provided ω is less than about 0.2, again a condition can usually be met in the field. In Figure 4, excellent agreement between C_2 of (17) subject to (18) and (39) subject to (37) are shown for various vertical distances ($0 \le \xi \le 1$) at two fixed radial locations of ρ equal to 0.02 (Figure 4(top)) and 0.1 (Figure 4(bottom)), where the decay process prevails. The same observations for the case without decay process are noted in Figure 5(top) ($\rho = 0.02$) and Figure 5(bottom) ($\rho = 0.1$). Differences between C_2 calculated from (17) and (39) become noticeable as ω is greater than about 0.2; Figure 6 indicates this condition for cases where ω is 0.3 and 0.4. Comparisons of C_1 from (18) together with (17) and from (37) coupled with (39) are shown in Figure 7 for both decay and nondecay cases and good agreement is noted.

Based on the above qualitative as well as quantitative discussions, it is concluded that the solutions of (17) and (18) are reasonable approximations for those of (37) and (39) if $l \le 0.001$ and $\omega \le 0.2$; both inequalities can oftenly be satisfied under field conditions. Therefore within the context of the assumed hydrogeological conditions, the proposed model provides a useful means in quantifying solute transport from an injection well into a leaky aquifer.

Contamination of the Unpumped Aquifer

In studying the problem of waste water injection into a leaky aquifer, one concern is to know whether the unpumped aquifer would be contaminated as the consequence of the injection. Although the proposed model does not investigate the solute migration in the unpumped aquifer, the solutions of C_2 evaluated at the top of the aquitard (i.e., $\xi = 1$ or $z = b_2$), nevertheless, is representative of the entering concentration level to the unpumped aquifer. Therefore $C_2(\rho, \xi = 1)$ can be employed to estimate the potential contamination conse-

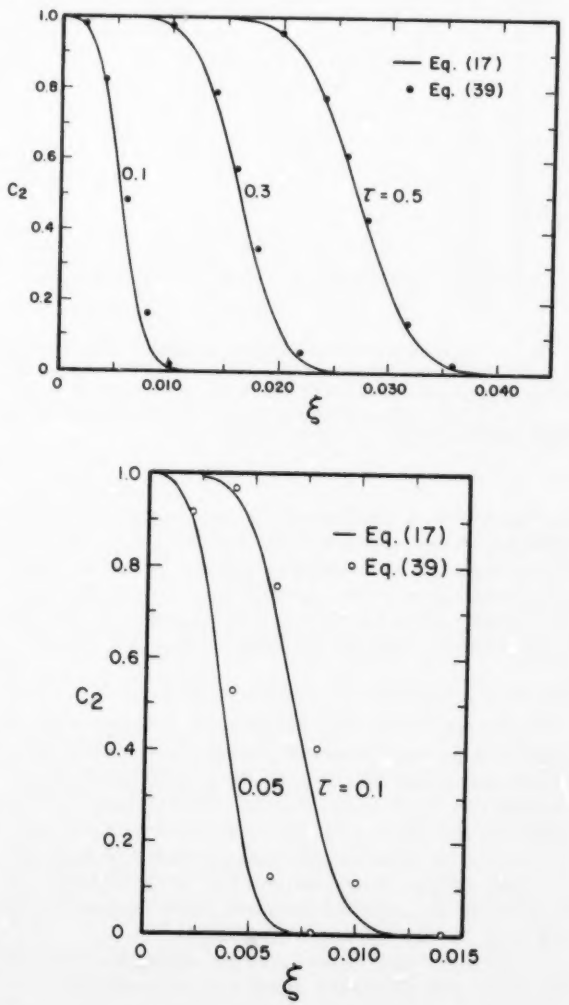

Fig. 6. Comparisons indicates the discrepancy between C_2 determined from (17) subject to (18) and C_2 determined from (39) subject to (37) as $\omega = 0.3$ and 0.4, both at $\rho = 0.02$ and $\alpha = \theta = 0$: (top) $\omega = 0.3$, $\eta = 1500$, and $\beta = 1.4 \times 10^5$ and (bottom) $\omega = 0.4$, $\eta = 2000$, and $\beta = 2.4 \times 10^5$.

quences in this layer. Considering C_1 and C_2 always attain their maximum values at ρ_0 due to the groundwater flow conditions involved, C_2 determined at ρ_0 and $\xi = 1$ represents the highest possible entering concentration to the unpumped aquifer. Under given hydrogeological and injection conditions, a higher $C_2(\rho_0, \xi = 1)$ always implies a higher possibility that the unpumped aquifer would be polluted. Type curves including all the necessary hydrogeological and injection parameters used in the proposed model are presented in Figure 8. Figure 8(top) pertains to conditions without any decay ($\theta = 0$) and Figure 8(bottom) gives results for various θ representing the decay conditions.

If no decay process takes place, Figure 8(top) shows that a small $C_2(\rho_0, \xi = 1)$ can result from a large β. If the aquitard thickness b_2 is large (and so is β) the vertical leakage velocity across the aquitard is small. In this event, the injected solutes primarily move through the pumped aquifer with a small portion leaking into the aquitard (thus a smaller C_2 at $\xi = 1$). If a large β is caused by a large R_2, obviously, a strong adsorption action exists in the aquitard which retards the solute movement from the pumped into the unpumped aquifer resulting in a small $C_2(\rho_0, \xi = 1)$. Figure 8(bottom) indicates that if θ is large (as could be caused by a strong radioactive decay),

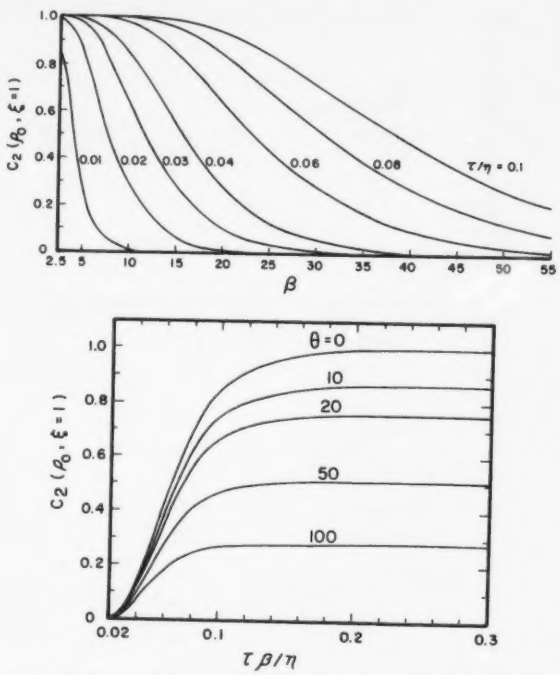

Fig. 8. Dimensionless type curves for $C_2(\rho_0, \xi = 1)$: (top) $\theta = 0$ and (bottom) θ is varied.

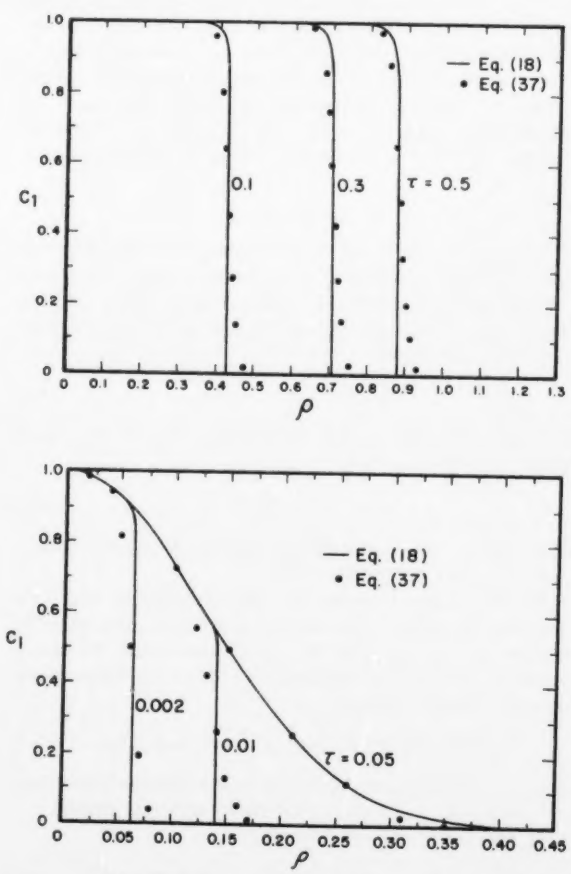

Fig. 7. Comparisons of semianalytical solutions of (18) subject to (17) with numerical solutions of (37) subject to (39); $\eta = 1000$, $\beta = 6 \times 10^4$, and $\omega = 0.2$. (Top) The case without the decay process with $\alpha = \theta = 0$ and (bottom) the case with the decay process with $\alpha = 60$ and $\theta = 3.6 \times 10^6$.

$G_2(\rho_0, \xi = 1)$ can be small due to the fact that most solute mass associated with the injection have been decayed before they can reach the top of the aquitard. Other possible physical interpretations of those type curves in Figure 8 can be made by examining the definitions of involved parameters in a straightforwa d manner.

Effects of Leakage

Chen [1986] gave an analytical solution for radionuclide transport from an injection well into a fracture contained in porous rock. The injected solutes can move by molecular diffusion into porous rock where groundwater is assumed to be immobile. The analytical solution given by *Chen* [1986, equation 59] assumes advection to be the only transport mechanism in the fracture (the pumped aquifer), which is comparable to the assumption used in the proposed model. Thus comparisons of this solution with the semianalytical solutions for C_1 of the proposed model can indicate the effects of groundwater leakage on the transport problem. In the comparisons, the pumped aquifer porosity needs to be included in Chen solution because the fracture porosity was assumed to be unity in the original work. Based on hypothetical data such as $Q = 8000$ m^3/day, $b_1 = b_2 = 10$ m, $n_1 = n_2 = 0.3$, $R_1 = R_2 = 1$, and $B = 100$ m, comparisons of these two different solutions are shown in Figure 9. Figure 9(top) refers to the case where the injected solutes are nondecaying ($\lambda = 0$). The solute front associated with the diffusion loss case is very steep reflecting the nature of advection dominant transport. A less steep solute front exists for the current study and it lags behind that from Chen solution (the diffusion loss case). The area between these two solute fronts are related to the amount of mass losses due to the leakage effects. In Figure 9(middle), the solutes are assumed to be radioactive with a half life of 27.8 days (i.e., $\lambda = 0.025$/day) as being for the radioactive tracer of $^{51}C_r$.

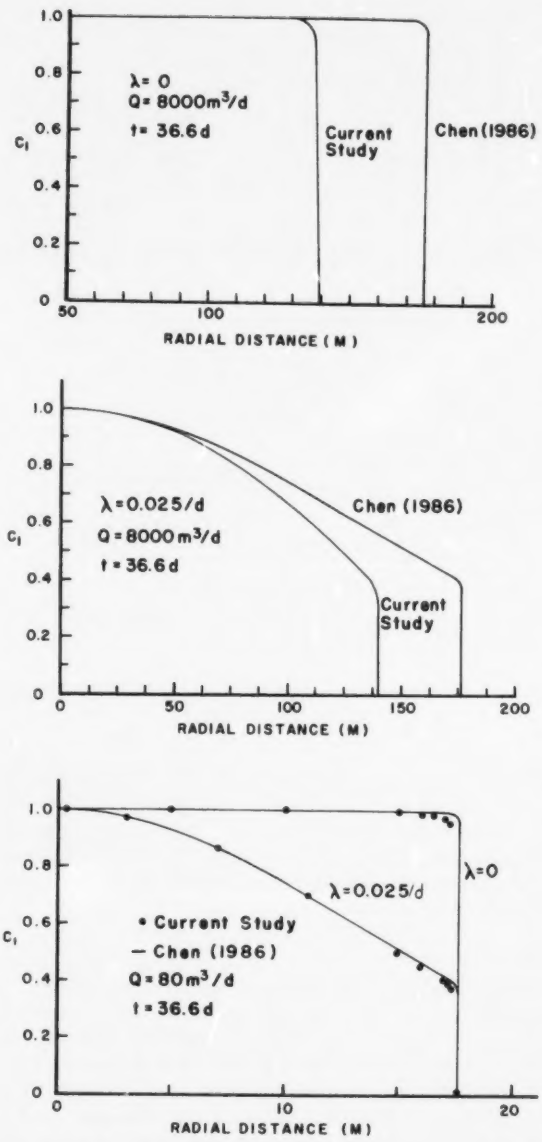

Fig. 9. Effects of groundwater leakage on solute transport in the pumped aquifer; the solution given by *Chen* [1986] neglects groundwater leakage effects while including molecular diffusion losses to the aquitard. (Top) The injected solutes are nondecaying with $\lambda = 0$, (middle) the injected solutes are radioactive decay with $\lambda = 0.025$/day, and (bottom) the pumping rate is reduced to 80 m^3/day and the pumped aquifer performs like a confined one.

The solute front subject to groundwater leakage after 36.6 days of injection significantly lags behind that subject to molecular diffusion losses. Therefore if the aquifer is of the leaky type (i.e., the leakage factor B is relatively small and the pumping rate is large) neglecting the leakage effects can introduce significant error in studying the transport problem. However, if the pumping rate is reduced to 80 m^3/day, the semianalytical solutions of the proposed model become almost identical to those determined with Chen solution (see Figure 9(bottom)). This confirms the earlier discussion that an aquifer subject to a small leakage factor may still perform like a confined aquifer when the pumping rate is small.

Summary and Conclusions

A mathematical model has been developed for the investigation of solute transport from an injection well into a leaky aquifer system, which consists of a pumped and an unpumped aquifer connected by an aquitard. In this system, the steady state groundwater flow field is strictly horizontal in the pumped aquifer, vertical in the aquitard and hydrostatic in the unpumped aquifer. On such a quasi three-dimensional flow field, injected solutes are assumed to be transported by radial advection in the pumped aquifer and by vertical advection and longitudinal mechanical dispersion in the aquitard. The unpumped aquifer is taken into account as a boundary condition to the aquitard. Two coupled linear differential equations are formulated for the transport problems in the pumped aquifer and in the aquitard; they involve coefficients dependent on Bessel functions due to the complexity associated with the groundwater flow conditions. Semianalytical transient solutions of the proposed model can be obtained by numerically inverting the Laplace domain solutions of the model with the method given by *Crump* [1976], of which a FORTRAN subroutine is readily available for use [see *International Mathematics and Statistics, Library, Inc.*, 1987]. Closed-form solutions for the steady state conditions are presented; they are nontrivial if the decay process exists. As compared to numerical solutions including longitudinal dispersion in the pumped aquifer and lateral dispersion in the aquitard, the modelling approach neglecting these two transport mechanisms has been proved to be reasonable provided $l \leq 0.001$ and $\omega \leq 0.2$. Since both of these inequalities can oftenly be satisfied under field conditions, the proposed model is considered to be a valid approximation to the problem of interest.

Appendix

This appendix discusses the influences of using two other possible boundary conditions at b_2 on the solution of C_2. One is related to the *Danckwerts* [1953] condition, which specifies a zero concentration gradient across the outlet and is written for the current study as

$$\partial C_2/\partial z = 0 \qquad \text{at } b_2 \tag{A1}$$

The other pertains to the assumption that a zero concentration always exists at b_2,

$$C_2 = 0 \tag{A2}$$

which implies that the unpumped aquifer dilutes the solutes arriving at b_2 to a negligible level.

The influences of (A1) and (A2) on the modeling results for C_2 can be discussed in the Laplace domain by examining the solutions of (24) subject to the two different outlet boundary conditions. The general solution of (24) for the other given boundary condition (28) is

$$G_2(\xi, \rho) = e^{\eta\xi}[G_1 \cosh(q\xi) + A_1 \sinh(q\xi)] \tag{A3}$$

where A_1 is a coefficient dependent on the boundary condition prescribed at b_2, and η and q have been previously defined.

The solution of (A3) subject to (A1) is

$$G_2(\xi, p) = G_1 e^{\eta\xi}\left[\cosh(q\xi) - \sinh(q\xi)\frac{q + \eta \coth(q)}{\eta + q \coth(q)}\right] \tag{A4}$$

The solution of (A3) subject to (A2) is

$$G_2(\xi, p) = G_1 e^{\eta\xi}[\cosh(q\xi) - \coth(q)\sinh(q\xi)] \tag{A5}$$

It is known that $\coth(q)$ approaches unity as q is about greater than 2.8 [*Abramowitz and Stegun*, 1970]. Therefore when $q \geq 2.8$, both (A4) and (A5) are reduced to

$$G_2(\xi, \rho) = G_1 e^{\eta\xi}[\cosh(q\xi) - \sinh(q\xi)] = G_1 \exp[(\eta - q)\xi] \tag{A6}$$

which is identical to (30) that is the solution of (24) subject to the outlet boundary condition at b_2 being extended to infinity. By noting the definition of q as given by (31), a sufficient condition for $q \geq 2.8$ is

$$\eta = b_2/2d_2 \geq 2.8 \tag{A7}$$

or

$$b_2 \geq 5.6d_2 \tag{A8}$$

Clearly, the inequality of (A8) can normally be satisfied under field conditions. Therefore regardless of the boundary conditions at b_2 being (29), (A1), or (A2), a unique solution for G_2 as given by (30) can be obtained provided $b_2 \geq 5.6\, d_2$.

NOTATION

Dimensional Symbols

- b_i uniform thickness, $i = 1$ and 2, m.
- B leakage factor defined in (4), m.
- C_i solute concentration, $i = 1$, 2, and 3, kg/m^3.
- d_i constant longitudinal dispersivity, $i = 1$ and 2, m.
- d_2' constant transverse dispersivity of the aquitard, m.
- D_2 dispersion coefficient of the aquitard, m^2/s.
- h_i head buildup, $i = 1$ and 2, m.
- k_i hydraulic conductivity, $i = 1$ and 2, m/s.
- Q constant injection flow rate, m^3/s.
- r radial distance, m.
- r_0 well radius, m.
- t injection time, s.
- T_1, T_3 transmissivity of the pumped and unpumped aquifer, m^2/s, respectively.
- u horizontal groundwater seepage velocity in the pumped aquifer as defined by (5), m/s.
- u' horizontal groundwater seepage velocity in a confined aquifer as defined by (1), m/s.
- v vertical groundwater seepage velocity in the aquitard as defined by (6), m/s.
- z vertical distance, m.
- λ radioactive decay constant, s^{-1}.

Dimensionless Symbols

- n_i porosity.
- q $[\eta^2 + (\theta + p\beta)/K_0(\rho)]^{1/2}$
- q' $[\eta^2 + \theta/K_0(\rho)]^{1/2}$
- R_i retardation factor.
- s_2 storage coefficient of the aquitard.
- α $2\pi n_1 b_1 R_1 \lambda B^2/Q$.
- β $n_2 R_2 b_2{}^2/(n_1 R_1 b_1 d_2)$.
- τ $Qt/(2\pi n_1 b_1 R_1 B^2)$.
- ρ r/B as the dimensionless radial distance.
- ρ_0 r_0/B as the dimensionless well radius.
- ξ z/b_2 as the dimensionless vertical distance.
- ω $d_2' b_2{}^2/d_2 B^2$.
- η $b_2/(2d_2)$.
- θ $2\pi n_2 \lambda R_2 b_2{}^2 B^2/(Qd_2)$.
- l d_1/B.

Subscript i

- 1 denoting the pumped aquifer.
- 2 denoting the aquitard.

Functions

- $K_0(\ ), K_1(\ )$ modified Bessel functions of the second kind of order zero and one, respectively.

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